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HANS FREUDENTHAL:

PERSPECTIVES IN MATHEMATICS EDUCATION AND PHENOMENOLOGICAL ANALYSIS

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Index

Introduction

For Freudenthal, the practice of mathematics in the curriculum is not a set of predetermined theories, goals and means. On the contrary, it is always associated with positively understood phenomenological processes in mathematics, as the curriculum is often used in conjunction with the transformation or development of practice. For Freudenthal (1968), educational theory was a practical endeavour from which new theoretical ideas could emerge as a kind of scientific by-product. In his view, curriculum development should not take place from the academic ivory tower, but in schools, in collaboration with teachers and students. He expressed similar ideas and, by calling for a practical curriculum, eloquently questioned curriculum theory in research and development and innovation.

There are, therefore, similarities between some of the implications of a colonial approach to curriculum and the concept itself. However, when phenomenology appears in Freudenthal's work, it often has the opposite meaning and he qualified the main direction of the movement as theoretical and top-down, starting from a vision of curriculum as processes and proposing an alternative of its own for educational development, which was nothing more than a perspective of mathematics education, that is, its realist mathematics (Sepúlveda, 2018).

From this perspective, phenomenological, curricular, and ethnographic real mathematics, or practice-based "mathematization" have similar characteristics. And this was of fundamental importance to Freudenthal because the main task of mathematics education should be the mathematization of everyday reality. Mathematics cannot be learned in mathematics courses because initially none of mathematics is empirically real for students, especially in the early years of life, i.e. in early childhood and primary education. In addition, the subject of mathematics, based on facts, helps students to adopt a mathematical approach in everyday situations (Trujillo, 2017). In this book, we can refer to the realistic mathematical activity, as expressed by Freudenthal (1991), which implies a mathematical attitude that includes the knowledge of the possibilities and limitations of learning a mathematical method, that is, the knowledge of when a mathematical method is appropriate and when it is appropriate or not.

This emphasis on "mathematization" is consistent with the need for mathematics for all. Freudenthal points out that not all students will become future mathematicians, since for most, all the mathematical knowledge they use will be used to solve everyday problems, that is, in the sociocultural context that surrounds them. Therefore, introducing students to mathematical methods to solve these types of problems deserves to be considered one of the main priorities of mathematics education, this is its phenomenology.

The story Freudenthal developed is better known for its critique of traditional educational research than for its own counter-empirical ideas and theories (Alsina, 2009). Freudenthal's opposition to much of educational research arose from his belief that deschooling was necessary, in the authors' view, a curriculum based on ethnography and lived experience. These breaks can be seen as method creation or the adoption of perspectives from different angles and, he argues, from this gap one can tell whether the student has reached a certain level of understanding or not. To identify these breaches, students must be observed individually. This means that group measures and the like are not particularly useful because they fill individual gaps. Furthermore, the emphasis should be on observing the learning process rather than examining objective learning outcomes.

Furthermore, Freudenthal believes that such extensive research cannot answer educational questions about why and who teaches a given subject. Freudenthal offers a second set of criticisms of the assessment movement. In theory, Freudenthal relied on formative rather than punitive education and focused on unfamiliarity with the subject matter and an overemphasis on reliability at the expense of validity.

Chapter 1

Freudenthal and phenomenological analysis

At the beginning of his work, Freudenthal introduces the term "phenomenology" to describe his method of analyzing mathematical concepts. He explains that this term is derived from the distinction made in the philosophical tradition between "phenomenon" and "noumenon." In mathematics, this distinction is seen between concepts or structures (noumena) and the phenomena that organize these concepts. For example, geometric figures organize the phenomena of contours.

Phenomenological analysis of a concept or structure involves describing the phenomena it organizes and examining the relationship between the concept or structure and these phenomena. This analysis must consider the development and current use of mathematics, as well as the original purpose of the concept or structure and its later extensions (Puig, 1997). The relationship between phenomena and concepts becomes more complex when mental objects are introduced, and phenomenological analysis must also consider the relationships between phenomena and mental objects, and between mental objects and concepts.

To better understand Freudenthal's ideas, it is necessary to strip the terms of their philosophical baggage. Instead of using "noumenon," it is more appropriate to refer to the concept as a "medium of organization." This emphasizes the function of concepts in relation to phenomena. The term "phenomenon" can still be used to describe what we experience mathematically, but its original meaning of appearance must be set aside. It is important to recognize that the means of organization of the phenomena we use to understand our mathematical experience are also an object of experience. The pair of phenomena and means of organization is defined by their relationship rather than by their belonging to different worlds.

This understanding allows us to view the series of phenomena and means of organization as interconnected, with the means of organization of one pair becoming the phenomena of the next. To engage in phenomenology is thus to describe one of these series or one of their pairs (Chakravorty, 2010). Although the terms 'phenomenon' and 'noumenon' are taken from the philosophical tradition, Freudenthal does not provide a clear definition of these terms. He claims that he does not use them in the sense given to them by philosophers such as Hegel, Husserl, or Heidegger, but he does not align himself with any specific philosophical system. The 'noumenon' is identified as the 'object of thought' without further explanation, and the 'phenomenon' is described as something we have experience of. These terms have Greek origins, where 'noumenon' means 'that which is thought by reason' or 'that which is intelligible', and 'phenomenon' means 'that which appears'.

Phenomena are the appearances or what we perceive of things, while noumena are considered the true reality in the realist philosophical tradition (Navarro, 1971). Identifying mathematical concepts with noumena places them outside our realm of experience. However, this contradicts one of the characteristics of mathematics pointed out by Freudenthal: that mathematical concepts are part of the field of phenomena that they organize. Mathematical concepts are not separate from our experience nor in a different world from that of the phenomena that they organize.

The phenomenological analysis developed by Freudenthal serves the purpose of didactics, although it is based on concepts from philosophy and has implications for the understanding of mathematics. Freudenthal distinguishes several types of phenomenology, all of which are important for didactics, but only one is specifically qualified as didactic. These types include phenomenology, didactic phenomenology, genetic phenomenology, and historical phenomenology.

Each type of analysis focuses on different phenomena in relation to the concept being studied. In pure phenomenology, the focus is on the current state and use of mathematics. In didactic phenomenology, the focus is on phenomena present in the students' world and those introduced in teaching sequences. In genetic phenomenology, the focus is on phenomena related to the cognitive development of students. In historical phenomenology, the focus is on phenomena that led to the creation and extension of the concept in question.

The description of the relationships between phenomena and the concept varies depending on the type of phenomenological analysis. In pure phenomenology, mathematical concepts are treated as cognitive products, while in didactic phenomenology they are treated as cognitive processes within the educational system (Zolkower and Bressan, 2012). Freudenthal emphasizes that a didactic phenomenology should not necessarily be based on a genetic phenomenology. The correct order for conducting diverse types of phenomenological analysis begins with pure phenomenology, followed by historical phenomenology, then didactic phenomenology, and finally genetic phenomenology. It is essential that any effective phenomenological analysis in teaching is supported by a sound analysis of pure phenomenology.

The nature of mathematics

Mathematical concepts are means of organizing phenomena in the world. However, this characterization lacks specificity unless we define what we mean by "the world" and identify the phenomena that mathematical concepts organize. Phenomenology, as a method, aims to investigate phenomena organized by mathematical concepts through analysis, rather than assuming prior knowledge of these phenomena.

Therefore, it is not possible to predetermine the type of phenomena organized by mathematics without conducting specific analyses. Freudenthal's phenomenological analysis serves as a foundation for the organization of mathematics teaching, but it does not go into depth in explaining the nature of mathematics itself. One could use this analysis without committing to any epistemological or ontological stance towards mathematics. In other words, one could view mathematical concepts as tools for organizing phenomena in teaching, without necessarily believing that this organization reflects the true nature of things. However, it is important to note that students' and teachers' beliefs about the nature of mathematics influence how they perceive mathematical activities in the classroom and the knowledge they aim to convey.

On the other hand, one could interpret from the above statement that mathematics exists in a world separate from the one it organizes, namely the real world around us. However, I do not find this interpretation to be the most adequate. In fact, if we consider the origin or the lowest level, the phenomena organized by mathematical concepts are those of the real, physical, everyday world. Our experiences with this physical world involve objects, their properties, the actions we perform on them, and the properties of those actions. Thus, the phenomena organized by mathematics are the objects, properties, actions, and properties of actions in the world, seen as means of organization and considered in relation to them.

The phenomena that organize mathematical concepts are the phenomena of the world that encompass human cognitive products, including the products of mathematical activity (Vergnaud, 1988). These phenomena include the objects of the world, their properties, the actions we perform on them, and the properties of those actions, in their relation to the means of organization. The gradual progression of phenomena and means of organization involves two processes: the creation of mathematical concepts as means of organization and the objectification of the means of organization, allowing them to form part of new pairs in the position of phenomena. This gradual progression illustrates the production of increasingly abstract and higher-level mathematical objects, demonstrating that mathematical activity generates its own content.

The first interpretation makes it clear that mathematical concepts do not exist in an ideal world that we study, nor do they have a pre-existing existence prior to mathematical activity. Mathematical activity is not about discovering the geography of a world where objects reside. However, it is also important to note that Freudenthal's analysis does not simply describe mathematical activity as an interplay between world phenomena and mathematical means of organization. Rather, Freudenthal recognizes that the means of organization themselves become objects situated in a field of phenomena.

Mathematical objects thus become part of our experiential world, entering as phenomena into a new relationship with other phenomena and means of organization. This process of incorporating mathematical objects into our world leads to the creation of new mathematical concepts, and this cycle continues repeatedly. Mathematics thus exists within the same world as the phenomena it organizes, and there is not a separate world but a unique world that expands with each mathematical product. It was mentioned earlier that the reason mathematical concepts do not exist as ideal objects in a separate world is due to the role of the sign systems in which mathematics is expressed or written.

The signs

Mathematical texts, regardless of the reader's experience, often contain symbols and terms that are not commonly used in everyday language. This distinction has led to the notion that the language of mathematics is separate from the vernacular. Furthermore, throughout the history of mathematics, there have been instances where new concepts and ideas were developed, further highlighting the unique nature of mathematical discourse.

The language of mathematics plays a crucial role and is often considered separate from everyday language. Some mathematicians believe that true mathematics is only written in a fully formalized language, while everyday mathematical texts are considered to contain "abuses of language" due to the use of the vernacular. However, rather than focusing on the types of signs used in mathematics, it is more important to study the processes of meaning and the production of meaning. Therefore, there is no need to separate mathematical signs from the vernacular or other means of representation.

Instead, all signs used in mathematical activity can be considered part of a mathematical sign system. These signs are not homogeneous and can be described using terminology borrowed from semiotics, which focuses on the expression and content of a sign. Mathematical sign systems contain signs with different expressions, like other human activities such as film or singing.

Mathematical concepts are created through the relationship between phenomena and the means of organization described by mathematical sign systems. These mathematical objects have a material existence that is given to them by the sign systems that describe and create them. As the means of organization become more abstract, mathematical sign systems also become more abstract and create corresponding abstract concepts (Streefland, 1991).

In the preceding discussion, I have highlighted the crucial elements of a mathematical framework that incorporates principles derived from Freudenthal's phenomenology of mathematics and Filloy's concept of mathematical sign systems. However, it is important to recognize that this description still lacks complexity. To provide a more complete understanding, I will now introduce additional ideas that I believe are essential for a complete description. These ideas are derived from the works of Lakatos and Kitcher, and while I will not delve into their complexities here, I encourage readers to consult the corresponding texts in the bibliography for further exploration.

In the realm of mathematical activity, the objects of experience consist of objects in the world, properties of the objects, actions performed on those objects, and properties of those actions. Keitel (1987) introduces the notion that the actions mentioned are not necessarily actions that we perform or can perform, but rather are actions that can be performed by an ideal subject endowed with greater powers of action than our own.

For example, this ideal subject might run through the sequence of natural numbers or use Hilbert's choice function. While this idea may suggest that mathematics can develop based on arbitrary stipulations of these powers, thus generating mathematical concepts that lack any epistemic or practical purpose, this danger is avoided in practice through various means.

While mathematical concepts are created in the process of organizing phenomena, they are not immutable once created. Rather, they are subject to modification over time due to their use and the new mathematical sign systems in which they are described. However, these modifications should not be seen as indicative of errors in the original concepts or as a linear progression toward a singular truth. This is because we reject the notion that mathematical objects have an existence prior to the process that creates them.

For Beltrán (1996), a unique perspective on the evolution of concepts in history is presented by Lakatos, who examines how concepts evolve under the pressure of proving the theorems in which they are involved. Lakatos highlights how the establishment of the $C + V = A + 2$ conjecture for any polyhedron and its proof by Euler led to the emergence of examples of solids that did not align with the proof or, more significantly, with the concept itself.

One way to address this danger is through the role of mathematical sign systems in the creation of mathematical concepts. These systems not only allow us to organize phenomena by creating relevant concepts, but they also allow us to perform new actions on mathematical objects or to appreciate the possibility of doing so if certain constraints were removed. These new actions are not arbitrary but are suggested by the more abstract mathematical sign systems, thus extending actions that we had previously seen ourselves capable of performing or that we had established as feasible at a lower level. Furthermore, the acceptance of these new actions by the mathematical community serves as a regulatory mechanism. However, this acceptance is not without controversy, as seen in the case of Hilbert's choice function.

According to one concept of mathematical objects, there exists an ideal object known as a polyhedron, and the purpose of mathematical activity is to discover its properties. This theorem provides evidence that the solids in question are not true polyhedral, or alternatively, the proof of their properties is flawed. However, it is important to note that Lakatos's interpretation of the story differs from this theorem. Lakatos distinguishes between two types of counterexamples: local and global counterexamples.

A local counterexample refers to a solid that has features that make the proof inapplicable, but still satisfies the relation in question. These counterexamples do not refute the conjecture; rather, they indicate that the proof is based on a property that was assumed to hold for all polyhedral, but this assumption is incorrect. Therefore, what is refuted is a slogan that has been used implicitly, thereby invalidating the proof. The existence of these counterexamples introduces a discrepancy in the concepts that was not present before.

The impact of the emergence of global counterexamples is important for our analysis. When a counterexample refutes a conjecture on a global scale, it is particularly noteworthy. Lakatos first presents global counterexamples to Euler's theorem, including a solid formed by a cube with a cubic hole inside it and a solid formed by two tetrahedra joined by an edge or a vertex. He then introduces an even more intriguing case of a solid star, the verification or denial of the relation depending on whether its faces are considered star polygons or not.

The existence of these counterexamples creates tension between the concept, the theorem, and its proof. Several approaches can resolve this tension and impact the concept of a polyhedron. The most fundamental approaches include: 1) Exclusion of monsters. The presented counterexamples are considered unconventional examples of polyhedral, referred to as monsters. While their existence is possible, it is not desired. The possibility of their existence is determined by the definition of a polyhedron that is used, leading to the development of a new definition that explicitly excludes them to maintain the theorem. 2) Exclusion of exceptions. The presented counterexamples are considered valid examples of the concept, even though their existence was not anticipated when the conjecture was originally posed. To return to safer ground, the conjecture is modified by introducing a distinction within the concept that separates these examples. 3) Adjustment of monsters. The objects are examined from a distinct perspective, causing them to cease being counterexamples. This is exemplified by the two diverse ways of perceiving star polyhedral: composed of star polygons or not.

Regular polygon: For any isometry S, a non-invariant point A0 exhibits an orbit, which comprises the set of points an into which A0 transforms by powers Sn, where n spans all integers. Consequently, we can describe the polygon A0A1A2... generated in this way as regular. Since the possible values of n span negative integers, the sequence of points extends both forward and backward, and the polygon A0A1A2... should be more accurately described as: ...A-2A-1A0A1A2... This example is particularly noteworthy because the definitions of polygon and polyhedron each incorporate distinct approaches to addressing the tension we are examining.

It is evident that the definition of a polygon is developed by accepting new objects, explicitly expanding the content of the concept (which will later require further classification into several types of polygons). Polyhedron: A polyhedron is a finite collection of planar polygons, known as faces, together with all their edges and vertices, which satisfy the following three conditions: i) Each edge belongs to exactly two faces, and these faces are not in the same plane. ii) The faces that share a vertex form a single cycle, that is, their intersection by a sufficiently small sphere, centered on the common vertex, constitutes a single spherical polygon. iii) No proper subset of the faces satisfies condition i).

Regular polyhedron: For any polyhedron, we define a flag (A, AB, ABC...) as the geometric configuration composed of a vertex A, an edge AB containing that vertex, and a face ABC... containing that edge. A polyhedron is considered regular if its symmetry group exhibits transitivity within its flags. Despite the simplicity of these basic methods for dealing with tension, the concept of a polyhedron is significantly affected in all cases.

Whether counterexamples are recognized or excluded as instances of the concept, the semantic field is expanded. In one scenario, the content of the expression increases, effectively broadening the range of phenomena that the concept was initially intended to encompass; this expansion constitutes the semantic field. In the other scenario, the concept establishes connections with new objects that were previously distanced from it in the new definition, thus becoming an integral part of its content. The entire process is more complex, involving increasingly intricate or abstract mathematical systems of symbols into which concepts, originally expressed in simpler or less abstract mathematical systems, are translated.

As Lakatos states, the concepts generated through this process do not improve the original concepts; they are not specifications or generalizations of them. Instead, they transform the original concepts into something completely different: they create new concepts. This is precisely what I intend to emphasize: the result of Lakatos' process of tension between concepts, theorems, and proofs is not the delineation of the true concept of a polyhedron that corresponds to a preexisting ideal object, but the creation of new concepts.

In the definition of a polyhedron, each condition is intended to exclude certain objects. For example, condition i) prevents the inclusion of a solid formed by two tetrahedrons joined by an edge as a polyhedron, thus excluding monsters from the category. However, this same condition also prohibits the interpretation of a star polyhedron whose faces are not the corresponding star polygons, since this would result in faces that lie in the same plane.

Therefore, this condition does not allow the adjustment of that monster, and star polyhedral are only considered polyhedral when their faces are the corresponding star polygons. The extension of the concept of polyhedron lies in the relationships that each condition establishes with other mathematical objects, as well as its representation within the system of signs used.

Lakatos's idea suggests that mathematical concepts are not fixed in their original form. They undergo changes, driven by the tensions that arise from being involved in proving and disproving. However, mathematical activity goes beyond simple theorem proving. Problem solving is a key factor in the development of mathematics, which includes theorem proving and other activities.

Problem solving involves proving theorems in two ways:

- First, it encompasses theorem proving on a global scale, where all problems are considered as problems and a distinction is made between finding and proving problems.
- Second, problem solving involves proving theorems in the context of each problem.

In fact, what characterizes problem solving in mathematics is that the result obtained must be accompanied by a justification that verifies the conditions of the problem. This extends the scope in which concepts are influenced and modified beyond the mere demonstration of theorems, to include problem solving. Furthermore, the formulation of recent problems and the study of families of problems are also important aspects of problem solving that contribute to the creation of mathematical concepts.

Problem solving is not the only mathematical activity that generates concepts. The organization of results obtained through problem solving and theorem proving into a deductive system is another crucial aspect of mathematics. This systematic organization can take many forms, from local to global and from axiomatic to formalized. However, it is a fundamental component of mathematics as mathematicians have moved from accumulating results and techniques to developing comprehensive frameworks. In this process, the use of definitions in mathematics has also evolved. In mathematics, definitions serve not only to explain the meaning of terms to people, but also as links in deductive chains within the activities of organizing deductive systems.

The process of definition involves organizing the properties of a mathematical object by deduction. It focuses on identifying properties that can be used to form a deductive system, either at a local or global scale, in which the mathematical object can be included. It is important to note that highlighting specific properties to define a concept is not a neutral or innocent act. On the one hand, it suggests that the concept was originally created to organize the corresponding phenomena. On the other hand, it means that the content of the concept is now determined by the deductions made within the defined system. Like theorem proving, the process of definition also leads to the creation of new concepts.

Objects and the acquisition of concepts

In this section, a new idea proposed by Freudenthal will be presented that challenges the ideas I have presented so far. This idea revolves around the distinction between a mental object and a concept. I find this concept particularly important because it leads Freudenthal to adopt a didactic stance: the primary goal of education should be the development of mental objects, with the acquisition of concepts being of secondary importance. The position is especially significant in the context of compulsory schooling, as it raises questions about what mathematics can offer to the general population. However, it also has relevance for the phenomenological analysis of mathematical concepts, particularly within the framework of didactic phenomenology (Treffers, 1987).

It is crucial to remember that this analysis precedes the organization of teaching and is done with that goal in mind. The mental object/concept distinction that Freudenthal presents can be understood as the result of considering the individuals who conceive or use mathematics in contrast to mathematics as a discipline or a historically, socially, or culturally established body of knowledge. In the previous sections, we have focused primarily on mathematical concepts within the discipline itself, with limited consideration of specific individuals. And various mathematical concepts and their relationship to phenomena and means of organization have been analyzed. Reference has also been made to the objectification of these means of organization and how they are integrated into a higher-level relationship between phenomena. Furthermore, how concepts undergo transformations because of mathematical activities such as theorem proving, problem solving, organization in a deductive system, and the process of definition has been explored. Throughout all of this I have emphasized that mathematical concepts do not exist independently of the mathematical activity that gives rise to them.

For example, consider the concept of a dog. The mental object associated with this concept is not capable of barking, as it is merely a representation of the concept in the mind. In the era of “modern mathematics,” attempts were made to teach students the concept of numbers through the Cantorian construction of cardinals. Although, if these students had relied solely on the official curriculum, they would have left school without acquiring a true understanding of numbers. Instead, they needed to develop a mental object of numbers outside of what the educational programs intended to teach them.

The transcript presented was about an exceptional individual who possesses superior abilities and performs actions that exceed our own capabilities. To understand this concept, we need to explore the contrast between mental objects and mathematical concepts. In everyday language, we typically refer to a person's understanding of a concept as their "concept," or we may use the term "conception" to emphasize that it is their personal interpretation of the concept. However, for the purposes of this discussion, we will use the term "mental object" to highlight that it is a part or perspective of the concept that exists within the individual's mind.

Thus, mental objects and concepts can differ in their interpretations and applications. By exploring the various contexts in which numbers are used, we can gain a deeper understanding of the meanings they have in different situations. To illustrate the difference between mental objects and concepts, I will use the complex and multifaceted concept of numbers as an example. Rather than following Freudenthal's approach, we will describe it in semiotic terms. Numbers are not only used in mathematical activities or in the classroom; they are also used in a variety of everyday contexts. These contexts can include sequences, counting, cardinals, ordinals, measurements, labels, written digits, magic, and calculations. Although I will not go into the details of each context, I do want to emphasize that there are distinct characteristics and meanings associated with numbers in each situation.

The semantic field of "number" encompasses all uses of numbers in various contexts, defining their encyclopedic meaning. Understanding the specific context in which a number is used enables the reader or receiver of a message to interpret it correctly. However, individuals do not have access to the full range of number uses in a culture or episteme. Rather, they operate within their own personal semantic field, which evolves and produces meaning in situations that require new uses of numbers.

This personal semantic field aligns with what Freudenthal calls “mental object number.” Freudenthal suggests that educational systems should aim to develop students’ personal semantic fields to encompass a wide range of number uses, ensuring that they can correctly interpret any situation involving numbers. Mundane uses of numbers occur in diverse contexts where phenomena are organized using the concept of number, both in its original form and in its extended applications.

The concept of mental objects helps organize these phenomena, allowing individuals to engage in activities such as numeration. Mental objects are formed through chains of phenomena and means of organization, like concepts. While the mundane contexts of number use mentioned above represent lower levels, the phenomenological richness of numbers in secondary school requires consideration of additional contexts, including mathematized ones. This explanation provides an initial understanding of what a mental object is and how it is formed. However, Freudenthal's term could simply have been called something else.

It should be emphasized that this is not the only difference between mental objects and concepts, and I do not mean to suggest that the relationship between them is simply a matter of relating a part of the mental object's content to its whole. Before delving into other aspects of this relationship, however, it is important to note that this account serves as the basis for Freudenthal's position, which I mentioned earlier in this section. According to Freudenthal, the acquisition of concepts is a secondary educational goal that can be postponed until a solid mental object has been established.

The relationship between mental objects and concepts is more complex than the number example I just provided, as my explanation has focused solely on comparing the expansion of the semantic field of number and Peano's definition, neglecting centuries of history that have shaped both contexts in which number is used, which retain traces of its organization through number concepts, and Peano's definition. The concept of number is a topic that requires further clarification. To introduce a term that distinguishes it, it is important to explain what other meanings the term "concept" has and how it differs from what we have called "mental object."

We have already established that mental objects exist in people's minds, while concepts exist in the realm of mathematics. However, this distinction alone would not be sufficient if we believed that mental objects are simply reflections of concepts in people's minds. The relationship between mental objects and concepts is more nuanced than that. Let me explain it in terms of semiotics. The mental object of number has been equated with the personal semantic field, which is made up of the various uses of numbers in different contexts and the culturally established meanings associated with them.

Mathematical concepts of natural numbers, such as those developed by Peano, Cantor or Benacerraf, for example, are the result of a long historical process. In the previous sections, I have examined the creation and modification of these concepts. From the semiotic perspective I am employing here, any mathematical concept of number that we wish to examine appears as the result of a definitional process that incorporates it into a deductively organized system, extracting it from the broader semantic field.

For example, Peano's concept of natural numbers, particularly in its more modern versions, breaks down meaning from the sequence context and presents it as a series of axioms that comprehensively describe its components. On the other hand, the concept of natural number derived from Cantor's construction is tied to the cardinal context, as indicated by the name Cantor originally gave it. In this account, concepts are related to a portion of the mental object, since the process of defining them involves selecting specific aspects of the meaning that the mental object encompasses.

When considering the processes involved in the creation and modification of concepts in history, it is important to understand the relationship between the mental object formed from these contexts and the concept of number defined by Peano. This relationship goes beyond a simple part/whole dynamic. A well-constituted mental object is one that can account for all uses and phenomena in diverse contexts.

The goal of educational systems, as described by Freudenthal, is to establish these good mental objects. To fully understand a concept, it is necessary to examine how it has developed in mathematics and how it has been organized within a deductive system. The specific relationship between a mathematical concept and its corresponding mental object influences how the mental object is constituted in relation to the acquisition of the concept. The constituents of a good mental object are determined through phenomenological analysis of the concept in question.

The analysis that has led us to differentiate between mental objects and concepts is a didactic analysis, focused on examining mathematics and its structures in the context of education. We recognize that students develop mental objects, which are their own personal understanding and organization of knowledge, and we also recognize the socially and culturally established content that we want students to learn, which we refer to as concepts.

Our exploration of the contrast between mental objects and concepts is done specifically within the educational system. In the context of the story, mathematical concepts are not something that exists before our experiences but are created through the mathematical activity of mathematicians. In this sense, mathematical concepts are simply crystallizations of mental objects. It is even possible, as Freudenthal suggests, for mathematicians to work with a mental object for a long time before formalizing it into a concept, as was the case with the concept of continuity.

So, what does it mean to “turn a mental object into a concept” in this context? I now approach the distinction between mental objects and concepts differently than when I initially examined it. I have associated mental objects with individuals because they are formed through subjective experiences and serve as a way for individuals to make sense of and have control over their experiences.

On the other hand, concepts are associated with mathematics as a discipline, but they also serve as tools for organizing phenomena. In the school system, concepts are introduced to students before they have direct experiences with the corresponding phenomena. The goal of the system is for students to develop mental objects that align with established concepts and to have access to the valuable means of organizing phenomena that history has provided us.

The topic at hand is the school system and its role in the creation of new mathematical concepts. This discussion focuses on the analysis of the mental objects used by mathematicians to organize phenomena and define them conceptually within the field of mathematics. It is through mathematical activity that concepts are derived from these mental objects.

This idea is also reflected in Lakatos' work, "Proofs and Refutations," where he mentions that the concept of a polyhedron is assumed to be understood, despite not being explicitly defined. As mathematicians prove theorems and explore diverse ways of modifying concepts, they find counterexamples that challenge and shape their mental objects, leading to the creation of new concepts.

It is common for discrepancies to arise between the mental object and the concept derived from it. In such cases, one may realize that one's mental object was not as well-formed as initially believed, prompting one to modify or revise the conceptual definition. For example, Freudenthal highlights the concept of continuity as a case in which a mismatch occurs between the mental object and the concept created.

When the first explicit definition of continuity was introduced, numerous examples of continuous functions that had not previously been considered as such emerged. However, over time, successive generations of mathematicians become accustomed to these new and unconventional cases of continuity. They reshape their primitive mental object to align with the concept defined by the new functions. It is important to note that the primitive mental object remains essential to the advancement of mathematics and is not simply replaced by the concept. Instead, a new mental object is formed that encompasses the concept created by the definition, or at least is provisionally compatible with it.

The relationship between mental objects and concepts can vary greatly. Mental objects are used to organize phenomena and precede concepts. However, concepts do not replace mental objects but contribute to the creation of new mental objects that include or are compatible with them. Sometimes, there is a significant gap between a mental object and a concept, as with the mental object curve and the Jordan curve concept. In the field of topology, mental objects alone do not provide deep understanding and it is necessary to form concepts that involve more than just local organization. These concepts fall into a broader field of study.

Phenomena can be organized at a higher level by mental objects such as spaces and manifolds of arbitrary dimension. These mental objects are then transformed into concepts by further organizational processes and the creation of more abstract sign systems to describe them. This transformation process is evident when considering the idea of a mental object and the progressive ascent through the chain of phenomenon/medium pairs of organization. However, it is important to note that not all domains of mathematics require concepts to progress.

Elementary geometry, for example, can be organized using only mental objects, without the need for concepts. In this case, concepts can be formed through local organizations, where the distances between mental objects and concepts are resolved. For example, a rectangle can be understood as a mental object without the concept of a "square," but a local organization can introduce the concept of a square without conflict. However, there are also examples in geometry where the distance between the mental object and the concept is obvious.

This is the case for elementary concepts like point, line, and surface, which closely resemble objects in the physical world. Physical objects, such as specks or marks made with a sharp object, suggest mental objects that need to be delineated and turned into concepts. This process is evident in Euclid's definitions of point, line, and surface, where mental objects are transformed into concepts. It is worth noting that the path of cognitive development does not necessarily go from one dimension to three, but rather the opposite.

Surfaces are experienced before lines, and mental objects associated with lines are constituted through various phenomenological sources, such as surface edges, arrows, threads, paths, and cuts. The difference between phenomenological sources and Euclidean definitions highlights the distance between mental objects and concepts. This distance becomes even greater when one considers the mental object involved in the transition from one dimension to another, specifically in the dimension itself. The transformation from dimension to concept is achieved in topology, where there is a significant distance between the mental object and the concept.

However, there is also a new type of property that arises when the properties evident in the mental object are difficult to prove, such as the n-dimensional Cartesian product of n segments. This creates a significant gap between the mental object and the concept of dimension. The gap becomes even more pronounced when fractional dimensions are introduced, forcing us to discuss dimensions in a non-traditional sense.

When analyzing didactic phenomenology, it is crucial to consider pure phenomenology, recognizing that there are cases where the distance between mental object and concept is insurmountable. This is especially true in secondary school teaching. Understanding the formation of mental objects through teaching involves recognizing the various forms this distance can take.

Alongside the examples mentioned above, it is important to note that there are cases where certain components are essential for the concept but not relevant for the constitution of the mental object. For example, the comparison of unstructured sets is crucial for the concept of cardinal numbers, but it plays a minimal role in the formation of the mental object because real-life situations rarely involve unstructured sets. Likewise, structure is necessary for making comparisons rather than being eliminated. Furthermore, didactic phenomenology reveals that the phenomena organized by a concept can be so diverse that they constitute different mental objects depending on the field of phenomena being explored in teaching. To fully understand the concept, it is necessary to integrate these different mental objects into a unified whole.

When it comes to comparing plane figures in terms of area, there are different methods available. A direct comparison can be made if one figure is contained within another, or an indirect comparison can be made using transformations, congruences, and other area-preserving techniques. Alternatively, figures can be measured individually. Measurement can be performed by covering the figure with unit areas or by approximations using interior and exterior methods. In these cases, additivity of areas is used when combining disjoint plane figures, or convergence of areas is considered when using approximations. However, it is not clear whether these approaches lead to the same measurement result and proving their equivalence is not a simple task. Lengths, areas, and volumes are fundamental measurements in elementary geometry.

These concepts are crucial in the study of measurement. The process of measurement begins by comparing the qualities of objects. This comparison is achieved by establishing a unit and considering objects as entities that possess the quality being measured. For example, if an object can be described as "long," then length can be attributed to it. However, understanding length, area, and volume as concepts presents challenges due to the various approaches to defining the mental objects of area and volume. In addition, there are mental objects that only manifest themselves in a mathematical or mathematized context. Analytic geometry serves as an example in secondary education.

The use of coordinate systems for global localization led to the algebraization of geometry throughout history. The Cartesian coordinate system is effective for describing geometric figures, mechanical motions, and functions. Geometric properties are expressed algebraically through relations between coordinates, motions in time-dependent functions, and geometric applications within systems of functions with multiple variables. The characteristic phenomena of analytic geometry can only be explored within mathematized contexts, using the sign systems of algebraic expressions and Cartesian representations.

Chapter 2

Freudenthal: Fundamentals of Algebra and Geometry

The number

It is important to recognize the multifaceted nature of numbers and their various interpretations and applications to fully understand their meaning and implications. The development of numerical concepts in secondary school can only be achieved through the formation of solid mental objects, traversing the semantic field of "number." It is important to note that this process of mental object formation is not a one-time event that results in an immutable mental object.

The meanings of numbers can be modified by factors such as the use of negative numbers in the ordinal and sequence context, or decimal expressions in the measurement context. Furthermore, it is important to recognize that in the Secondary School numbers are not only used in the contexts mentioned above, but also in other mathematized contexts. Exploring the concept of "number" in natural language requires a didactic phenomenology that considers how mental objects are formed and integrated to account for the diverse uses of "number" in different contexts (Burkhardt, 1988).

This exploration begins outside of school and continues throughout the Primary stage. To introduce a phenomenological analysis relevant to the Secondary stage, it is necessary to make a brief reference to these initial phenomena and to study how the meaning of "number" is extended and modified by the new phenomena that students encounter at this stage. The extension of the meaning of numbers reaches its peak in what can be called algebraic access to the concept of number. This involves organizing arithmetic operations and constructing a higher-level concept of number that is distinct from those formed in previous contexts.

In this context, a number is defined as something that allows for arithmetic operations. This new concept of number, which has historical significance, legitimizes all numbers that were previously considered alien to the true concept of number and were consequently given different names. As mentioned above, the mental object of "number" is formed to organize various phenomena, which can be understood by examining the objects to which numbers refer (individual objects, sets, words, the numbers themselves). This includes considering whether the objects are discrete or continuous, whether they are ordered or not, and the nature of the units and the number that describes the objects. These characteristics can be categorized into different contexts of number usage, such as cardinal, ordinal, measurement, sequence, counting, label, magic, and reading contexts. The combination of all these usages in different contexts forms the semantic field of "number," and an individual's experiences contribute to the formation of his or her mental object of "number."

The concept of exclusion, which refers to the idea of considering non-numeric objects as numbers, has been around since the 11th century. During this time, Arab algebraists such as al-Karajī were already treating algebraic objects as numbers. This idea was further developed by Cantor, who aimed to show that the transfinite numbers he had introduced were in fact legitimate numbers.

The concept of “number” also applies to all basic arithmetic operations. Thus, addition and subtraction, their meanings are derived from the actions of counting, combining sets, and comparing sizes within a field of meaning that is integrated through corresponding teaching tools such as the number line. However, multiplication and division have even more diverse meanings. Unlike “number,” the meanings of these operations are influenced by new phenomena and diverse types of numbers. For example, when dealing with fractions or decimal numbers, we need to expand our understanding of “number of times” to understand multiplication. Furthermore, in more advanced mathematical contexts, operations are often seen as extensions of algebraic concepts.

A ratio is a mathematical function that relates two numbers or values. It can be calculated using basic arithmetic operations, but what matters is the value that the function assigns to each pair, which can be determined using algorithmic procedures. However, if we interpret a ratio as simply the result of a division, we lose sight of its true meaning. The importance of a ratio lies not in the process of assigning a value, but in the ability to compare ratios of equality or inequality, without knowing their specific magnitudes.

The concept of ratio allows us to express statements like "a is to b as c is to d" without reducing these comparisons to numerical values. From a phenomenological perspective, the logical status of ratio can be described in terms of the equivalence relation "having the same ratio." This is consistent with Euclid's definition in Book V of the Elements, where he does not define "ratio" itself, but rather the notion of ratios having the same ratio. The logical status of ratio is therefore considered to be at a higher level than that of numbers, fractions, lengths, and other concepts typically encountered in education. This higher level is characterized by an intensive property, which organizes the relationships between objects or sets of objects rather than focusing on their extensive properties.

The variety of intensive properties of objects organized by reason is enormous and encompasses a wide range of factors. In the context of teaching, it is important to consider an important division within these properties: the relation can be a relation within a single magnitude or between multiple magnitudes. This division can be represented by two measurement spaces or magnitudes, with a linear application connecting them.

The relationship within a magnitude is considered internal, while the relationship between the two magnitudes is considered external. A proportion involves a linear function that relates these measurement spaces. Linearity implies that the internal ratios remain constant under the function, and the external ratios between the elements to which the function corresponds are also constant.

Linearity is demonstrated through the implicit concept of equal spaces traversed in equal times for internal ratios, and through the explicit function $f(x)=\alpha x$, where α represents a constant, for external ratios. Furthermore, a didactic phenomenology reveals that the development of the ratio and proportion mental object involves precursor mental objects. These precursor mental objects are often qualitative in nature and involve comparisons of ratios, providing a context in which the concept of equality of ratios, or proportion, can be understood. An important precursor mental object, identified by Freudenthal as the "relatively" mental object, plays a crucial role in this process.

The concept of "relatively" allows for meaningful statements, such as that one chocolate is sweeter than another because of its higher sugar content. The term "relatively" refers to a criterion of comparison, which may be implicit or explicit, such as weight. The mental object of "relatively" is developed in teaching through several steps, including understanding that dispositions can be relativized, understanding the meaning of "relatively" as "in relation to...", using "relatively" and "in relation to" with understanding, providing appropriate completion of these terms within a given context, operational knowledge of their meaning, and the ability to explain their meaning to others.

Algebra

Modern algebra is a field of study that categorizes and analyzes the structural features of collections of various objects, incorporating defined operations within them. These properties and objects are derived from the way lower-level phenomena are organized and have evolved over time. A significant event in the history of algebra is the writing of the Concise Book of al-Jabr and al-Muqabala by al-Khwarizmi in the ninth century. This marks the birth of algebra as a distinct discipline within mathematics.

Al-Khwārizmī's approach to algebra is unique in that he establishes the diverse types of numbers needed for calculations, such as treasures, roots, and simple numbers. He then explores the combinations of these types and develops algorithms for solving each type. These types serve as canonical forms to which any problem can be reduced. Al-Khwārizmī's contribution lies not in the methods of solution, but in the establishment of a comprehensive set of solvable canonical forms and the organization of their application to problem solving.

Another significant advance in algebra is attributed to Galois, who moved from searching for novel solutions to studying the solvability conditions of equations. The history of algebra since then can be seen as a series of advances resulting from the objectification of earlier level organizational methods. However, this historical perspective is not particularly relevant to algebra taught in today's high school curriculum, as it has abandoned the modern algebra introduced in the 1970s.

Instead, it is more important to analyse the characteristics of natural language and the sign systems of school arithmetic, as they provide the basis for students to acquire the language of algebra. To carry out this analysis, Freudenthal examines various aspects, such as transformation rules in natural language, arithmetic language, language as action, formalization as a means and as an end, the algorithmic construction of proper names, punctuation rules, and the use of variables in everyday language.

In the language of mathematics, variables play a significant role. The equal sign is another crucial element that helps establish equality between different mathematical expressions. Algebraic strategies and tactics are employed to solve equations and manipulate mathematical expressions. Formal substitution is a technique used to replace variables with specific values.

The algebraic principle of permanence states that if two expressions are equal and one is modified, the other expression must also be modified in the same way to maintain the equality. Algebraic translation involves converting posed problems or real-life situations into algebraic expressions or equations. Moving on to linear algebra, it is considered a practical tool rather than a historical study of its origins.

Rather than exploring the historical context of linear algebra, the focus is on its applications and how it can organize and make sense of phenomena in various fields. The concept of linear algebra can be understood by examining its meaning in specific application contexts. For example, the product of matrices can be given significance by applying it to graph connectivity matrices. This understanding can then be extended to other contexts without relying on the specific content but on the expression and manipulation of matrices.

Geometric objects

The concept of space, both as a mental construct and as a mathematical idea, did not originate as a foundation of geometry. Rather, it is the result of a prolonged process of development. According to Freudenthal, while geometric objects exist within space as concepts, the corresponding mental objects associated with these concepts are situated within a geometric context.

Our starting point, both in the historical sense and in the personal history of everyone, is not phenomena that can only be experienced at a level already defined by mathematics and organized by the concept of space. Rather, it is other phenomena and contexts that come first. Initially, organized phenomena are shapes and configurations that are observed in a visual context, such as contours and lines of sight. These are closely related to human creation itself, as humans produce “geometric” shapes.

Geometrical objects, as concepts, develop from mental objects that serve as tools for organizing the "geometrical" figures that are observed or drawn on Earth. In defining these objects, their definitions must be separated from the sensory properties of the figures they are intended to organize. For example, Euclid defines a point as having no parts and a line as a length without width, using properties that highlight deficiencies and detachments. This creates a concept separate from the mental object that organizes the corresponding phenomena. Therefore, any action or observation made with these figures must be analyzed to be accepted as a geometrical object. For example, if a circle is drawn on the ground or on paper and a tangent line is added to it, in the drawing they do not intersect at a single point but "along their entire length," as Protagoras argues.

To learn geometry effectively, drawings must be placed in geometric contexts. Furthermore, the relationships between the drawing and the geometric object cannot be learned without proper instruction. Colette Laborde proposes a solution to this problem in the context of the Cabri-geometer. Students are exposed to problem situations involving drawings, where geometry becomes a useful tool for modelling and problem solving. For example, geometry allows the creation of drawings that meet specific constraints in a more efficient way than trial and error. The accuracy of the results can be ensured by geometric principles, such as the tangency of a line to a circle when it is perpendicular to the radius.

Likewise, students are also presented with geometry situations where drawing and experimentation help to avoid lengthy theoretical solutions.

The very nature of the drawing also influences its interpretation. A drawing can only refer to theoretical objects of geometry to the extent that the reader chooses to interpret it as such. The interpretation is influenced by the reader's chosen theory and his or her knowledge. And the context in which the drawing is presented is crucial in determining the type of interpretation. A well-developed mental object must incorporate an analysis of the elements of the figure and the relationships between them. Also, the relationship between the drawing and the geometric object is complex because it requires interpretation by a human subject.

Consequently, a geometric drawing may not always be interpreted as representing a geometric object, and interpretations of the same drawing may vary depending on the reader, their knowledge, and the context. During Euclid's time, mathematicians discovered that there was a significant gap between the concept constructed by Euclid and the primitive mental object. However, they also realized that geometric figures drawn on paper, known as geometric drawings, were used to represent geometric objects. This relationship between figure, drawing, and geometric object is crucial in the formation of corresponding mental objects and the understanding of concepts.

In contrast, when a computer environment such as Cabri-geometer is considered, the dynamics between drawing and geometric objects are altered, as we have just discussed. Drawings created with Cabri on a computer screen, known as Cabri-drawings, exhibit different behaviors compared to drawings made with pencil on paper. This is because Cabri drawings are not drawn manually but are defined by program primitives that always align with geometric properties.

The modification of a Cabri drawing resulting from the movement of its elements disrupts certain interpretations of the drawing based on its spatial properties. Moreover, in the experience of students, the scope of phenomena organized by geometric objects in Secondary School is incredibly diverse and can be found in various aspects of nature, art and human creations. Although we will not make an exhaustive list, it should be noted that this abundance of geometric phenomena has significantly expanded in recent times with the emergence of infographic products such as video games, digital images used as covers of television programs, music videos and computer games.

The movements

Geometric transformations involve the physical motion of geometric figures. However, there is a complex and conflicting relationship between the geometric characteristics of these transformations and the spatial properties of the motions. This leads to difficulties in recognizing that different paths can result in the same transformation, accepting certain transformations as identities. It is important to consider this gap between real-world phenomena and mathematical concepts of geometric transformations. There are many phenomena in the students' environment that can be explored and that are relevant to understanding geometric transformations.

Statistics

Descriptive statistics has been developed for the purpose of organizing numerical data provided and covers a wide range of social, political and economic phenomena. These phenomena serve as contexts in which statistical concepts are applied and it is important to consider them as educational opportunities, as they allow individuals to experiment and form mental objects related to statistics.

Statistics concepts are primarily concerned with quantitative information contained in data and aim to summarize, characterize, and organize it in a way that facilitates comparison with other massive data sets. In everyday life, statistical concepts can be observed in various media such as newspapers and television, where they are used to describe different topics or predict voter behavior during political campaigns.

These real-life applications contribute to the understanding and formation of mental objects related to statistics. However, within the realm of statistical inference, phenomena become more complex and abstract as they involve deriving knowledge from observing features of cases. Ian Hacking has pointed out the challenges of establishing statistical inference within the framework of Galilean science as it requires navigating the concept of "acceptable evidence" in different historical periods and social practices. Fisher's contribution to this field is significant as it introduces the idea that rejecting the null hypothesis is not equivalent to completely refuting it. Instead, the alternative is to reject or err in rejecting the null hypothesis.

The probability

Probability has its roots in the realm of gambling and uncertain events, such as lottery drawings or weather patterns. In everyday language, the term "likely" can have two meanings: it suggests that something may happen, but it also implies a personal belief that it will happen. The distinction between these interpretations is often conveyed through the speaker's emphasis. Thus, the terms "likely" and "possible" have a closely related meaning. Various perspectives, including logicism, subjectivist, and frequentist views, have been proposed to explain the concept of probability. These theories shed light on the diverse range of phenomena that fall under the umbrella of probability.

Challenges are often encountered in defining the fundamental ideas of probability and randomness. A solid understanding of randomness and probability can be built using elements derived from Kolmogorov specifications: An event may have multiple outcomes. The outcome of the event cannot be predicted with certainty, even with the knowledge of those observing it. The event may be repeated under identical conditions a substantial number of times, allowing generalizable conclusions to be drawn. The sequence of outcomes obtained during repetition does not show any discernible pattern that can be predicted by the observer. As the number of repetitions increases, fluctuations in the relative frequencies of outcomes become more stable and gradually decrease in magnitude in a predictable manner.

The variables

The current practice in mathematics of referring to "variables" as means of expressing general propositions, which are what Freudenthal calls "multi-purpose nouns," is a recent development. Traditionally, the term "variable" has always denoted something that fluctuates or changes, whether in the physical, social, mental, or mathematical realm. Initially, the concept of variables encompassed observable phenomena in the physical, social, and mental realms, but it was eventually expanded to include mathematical objects such as numbers, magnitudes, and points, which are also considered variables.

The concept of function originates from the recognition of a relationship or dependency between variables. This dependency may be established, postulated, generated, or replicated within the physical, social, or mental domains, as well as between mathematical variables that may also have connections to variables in other domains. As this dependency is further explored, it can be objectified and treated as a mental construct. Before this objectification can occur, however, the dependency must first be experienced, used, stimulated, made conscious, considered as an object, given a name, and situated within a larger framework of interdependencies:

- Functions arose as relationships between magnitudes of variables whose variability was compared in infinitesimal quantities.
- The freedom to change variables from dependent to independent and between independents led to a new kind of operation with functions: composition and inversion. It was this new operational richness that led to the success of the function concept.
- The need to distinguish between dependent and independent variables has led to an emphasis on functions rather than relationships. Despite what algebraic and analytical expressions suggest, development has tended toward univalent functions.

- A change in perspective led from describing visual data through analytically expressed functions to visualizing functions through graphs.
- An arbitrary function appears in the calculus of variations and solving differential equations. This "arbitrariness" refers not only to the nature of the functional dependence, but also to the nature of the variables, which can be numbers, points, curves, functions, elements of arbitrary sets.
- Analysis functions, geometric transformations, permutations of finite sets, and applications between arbitrary sets combine to create the general concept of a function.
- This concept, in turn, is used to organize a wide variety of objects, from algebraic operations to logical predicates.

This latter wealth of such diverse phenomena integrated into the general concept of function, phenomena which, moreover, many of them belong to the world of mathematics itself, makes function as a mental object much more complex than the number of objects, geometric or even reason. The concept of function can only be mastered at advanced stages of schooling when students may already have had experience with many these phenomena. What can really be constituted as a mental object in secondary education is the idea of variable and functional dependence, and it is difficult to overcome even the geometric transformations that are also being experienced.

By delving exhaustively into these concepts, which unfortunately cannot be fully explored within the confines of this discussion, we realize the vast gulf that exists between these concepts themselves and their origins in the initial phenomena and first mental constructs that emerge in the realm of mathematics and in the firsthand experiences of individuals. This is reminiscent of Cantor's famous comment to Dedekind, in which he confessed, intentionally, that he was able to perceive the validity of a certain concept, but that he struggled to fully embrace its reality.

The author's argument for the existence of infinitely different cardinals using the diagonal method serves as a clear example of the challenges posed by mathematical concepts such as continuity and infinity. These concepts have no direct correspondences in our physical experiences, and it is unclear whether infinity is a singular concept. Mathematicians developed these concepts to understand and organize phenomena that occur within the realm of mathematics, where objects are produced or occur in highly mathematized contexts. However, teaching these concepts requires considering that a deep understanding of them can only be achieved through experience of the phenomena they organize.

While the acquisition of these concepts may present difficulties, it does not mean that they should be abandoned. Instead, a teaching approach should involve creating a variety of experiences that encompass the various relevant phenomena and organizing instruction in a way that allows for the formation of a mental object capable of dealing with those phenomena. Furthermore, it is important to recognize that many of the phenomena crucial to forming strong mental objects are intrinsic to the mathematical means of organization themselves, which can be studied at a higher level.

Chapter 3

Perspectives on mathematics education

When considering one's ideas about mathematics, a variety of opinions and beliefs about the subject, the mathematical activity, and the ability to learn mathematics may emerge. This discussion may seem irrelevant to teachers who are primarily concerned with improving the effectiveness of their teaching. The concept of knowledge belongs to the branch of philosophy known as epistemology, which explores theories of knowledge. However, beliefs about the nature of mathematics can influence a teacher's actions in the classroom.

For example, imagine a teacher who believes that mathematical objects have an existence of their own, albeit immaterial. From this teacher's point of view, objects such as triangles, sums, fractions, and probabilities exist independently of the people who use them or the problems to which they are applied, and even independently of culture. According to this belief, the best way to teach mathematics would be to present these objects to students, much as one would show a child an elephant at the zoo or through a video.

How can we demonstrate what a circle or other mathematical object is? This teacher would say that the best approach is to teach the definitions and properties of these objects, as this is what constitutes "knowing math." Applying concepts or solving mathematical problems would be considered secondary and would be taught once the student has mastered the mathematical objects themselves. On the other hand, some teachers view mathematics as a product of human ingenuity and activity, much like music or literature. They view mathematics as something that was invented because of human curiosity and the need to solve a variety of problems, such as trading goods, construction, engineering, and astronomy.

For these professors, the fixed nature of mathematical objects today, or in earlier historical periods, is the result of social negotiation. The individuals who created these objects had to agree on their rules of operation, ensuring that each new concept fit coherently with existing ones. Moreover, the history of mathematics shows that the definitions, properties, and theorems established by renowned mathematicians are not infallible and are subject to evolution.

With this perspective in mind, learning and teaching mathematics must recognize that students naturally encounter difficulties and make mistakes during the learning process, and that these mistakes can serve as valuable learning opportunities. This aligns with constructivist psychological theories about mathematics learning, which are based on the philosophical view of mathematics known as social constructivism.

The Platonic idealist conception

Within the wide range of beliefs about the relationship between mathematics and its practical applications, as well as its importance in the field of teaching and learning, there are two distinct and contrasting conceptions. One of these conceptions, which prevailed among many professional mathematicians until recently, asserts that students must first understand the fundamental structures of mathematics in an axiomatic way.

It is assumed that once this foundation is firmly established, the student will find it easy to solve the various applications and problems that arise. According to this perspective, without a solid mathematical foundation, it will be difficult to apply mathematical principles except in extremely simplistic scenarios. Consequently, pure and applied mathematics are perceived as separate disciplines, with abstract mathematical structures taking priority over their applications in the natural and social realms.

Practical applications of mathematics are considered an "appendix" to the study of mathematics, and thus ignoring this appendix would not cause any harm to the student's understanding. Those who subscribe to this belief view mathematics as an autonomous discipline, capable of developing without considering its applications to other sciences, based solely on internal mathematical problems. This conception of mathematics is known as the "idealist-platonic" perspective. This perspective allows for a simple curriculum, as there is no need to incorporate applications from other areas (Gravemeijer, 1994). Instead, these applications are "filtered" to abstract away mathematical concepts, properties, and theorems, resulting in a "pure" mathematical domain.

The constructivist conception

Many mathematicians and math teachers argue that math should have a strong connection to its real-world applications in various subjects. They believe that it is crucial to demonstrate to students the practicality and relevance of each math concept before introducing it. The idea is to show students how distinct parts of math serve specific purposes. For example, by placing children in situations where they need to exchange items, we create a need for them to compare, count, and organize collections of objects.

This gradually introduces them to the concept of natural numbers. According to this perspective, applications of mathematics, both external and internal, should come before and after the development of mathematical concepts. These applications should emerge naturally in response to problems encountered in the physical, biological, and social environments in which humans live. Students should be able to recognize that axiomatization, generalization, and abstraction in mathematics are necessary to understand and solve real-world problems.

Proponents of this approach to mathematics and its teaching would prefer to start with problems from nature and society and use them as a basis for building the fundamental structures of mathematics. In this way, students would gain a deeper understanding of the close relationship between mathematics and its applications. However, developing a curriculum based on this constructivist approach is complex because it requires knowledge not only of mathematics but also of other fields.

Structures found in the physical, biological, and social sciences are often more complex than those found in mathematics and do not always align perfectly with purely mathematical structures. Although there is a wealth of material on the applications of mathematics in a variety of areas, the task of selecting, organizing, and integrating this material is not easy.

Mathematics and society

Teaching mathematics involves not only imparting knowledge and skills, but also fostering understanding and appreciation for the role of mathematics in society and the power and limitations of the mathematical method. Mathematics has a rich history of evolution to solve practical problems and has played a crucial role in the development of various fields. Thus, concepts within mathematics have also undergone changes over time, adapting to novel approaches and advances in technology.

By recognizing these aspects, educators can provide students with a comprehensive understanding of mathematics and its importance in society. This pattern of mathematics evolving in response to various problems is not unique to statistics. Geometry, for example, emerged from the need to solve agricultural and architectural problems. Different numbering systems evolved along with the need for faster arithmetic calculations.

Probability theory was developed to address problems related to gambling. Mathematics serves as a framework upon which scientific models are built, aiding in the process of modeling reality and often validating these models. Mathematical calculations, for example, allowed the discovery of the existence of distant planets in our solar system long before they could be observed. From a historical perspective, mathematics is a constantly evolving body of knowledge.

Throughout its evolution, the need to solve practical problems (both within mathematics itself and in relation to other disciplines) has played a significant role. For example, the origins of statistics can be traced back to ancient civilizations such as the Chinese, Sumerian and Egyptian, which collected data on population, goods and production. Even the Bible refers to counting Israelites of military service age in the book of Numbers. Censuses themselves were already established in the Roman Empire in the fourth century BC. However, it is only recently that statistics has acquired the status of a science.

In the 17th century, political arithmetic emerged from the German school of Conring. Achenwall, a disciple of Conring, focused his work on the collection and analysis of numerical data for specific purposes, laying the foundations of the statistical method. This example demonstrates how mathematics, including statistics, has developed in response to various problems.

When considering the teaching of mathematics, it is crucial to reflect on two important objectives:

- First, our goal is for students to understand and appreciate the role of mathematics in society, including its diverse applications and contributions to social development.
- Second, we strive to ensure that students understand and value the mathematical method, which encompasses the types of questions that mathematics can help answer, the fundamental ways of reasoning and working mathematically, as well as their strengths and limitations.

However, the evolution of mathematics is not only the accumulation of knowledge or the development of new fields of application. The concepts themselves within mathematics have undergone changes in meaning over time, being expanded, specified, revised, gaining relevance or, at times, being relegated to the background. For example, the calculation of probabilities underwent a significant transformation with the incorporation of concepts from set theory into Kolmogorov's axiomatics.

This innovative approach allowed the application of mathematical analysis to probability, leading to advances in the theory and its practical applications in the last century. Similarly, the manual calculation of logarithms and circular functions (e.g., sines, cosines) used to be widely taught, with students spending hours learning related algorithms. However, with the advent of calculators and computers, these functions can now be calculated directly, making manual calculation obsolete. The same trend is occurring today with the calculation of square roots.

The purpose of mathematics

Mathematical applications play a vital role in our everyday lives, and it is critical that students recognize and appreciate this. To achieve this, it is important for us to provide comprehensive examples and situations in class that demonstrate the wide range of phenomena that mathematics allows us to organize and understand (Higginson, 1980).

Beyond the biological and physical context of the individual, we live in a society filled with mathematical situations. Family, school, work and leisure activities all involve mathematical elements. Numerical or statistical studies can be conducted to quantify the number of children in a family, the age of parents at marriage, the types of work people do and the different beliefs and hobbies within different families. Public transport is relied upon to get to school or go on holiday, and we can estimate factors such as travel time, distance and the number of passengers using buses.

During our free time, we participate in games of chance such as billiards or lotteries, and we attend sporting events whose outcomes are uncertain, and we may have to queue to buy tickets. Buying insurance policies, investing in the stock market and making financial decisions are examples where statistics and probability are essential tools.

One area where mathematics is applicable is biology. Students can be reminded that many inherited characteristics, such as sex, hair color, and birth weight, cannot be predicted in advance. Also, traits such as height, heart rate, and red blood cell count can vary depending on when they are measured. Probability allows us to describe and analyze these characteristics.

In the field of medicine, statistical epidemiological studies are performed to quantify a patient's condition and track its progression using charts and graphs. These conditions are then compared to average values in a healthy individual. Determining the red blood cell count from a blood sample is an example of situations involving proportional reasoning and the concept of sampling.

Mathematical models of population growth are also used to make predictions about world population, the possibility of whale extinction, the spread of diseases, and even the life expectancy of individuals. Nature provides us with numerous examples of geometric concepts, which can be abstracted and studied. As students grow older, measurement activities can be proposed to help them differentiate different magnitudes and estimate quantities such as weight and length. Governments at various levels need information to make informed decisions. That is why censuses and surveys are conducted. From election results to population censuses, a wide range of statistics influence government decisions. Consumer price indices, labor force rates, immigration and emigration, demographic statistics, and the production of different goods and trade are all examples of ratios and proportions that we hear about daily in the news.

In the economic world, national and corporate accounting, as well as the control and forecasting of production processes, are based on mathematical methods and models. In our complex economy, it is essential to have a basic understanding of financial mathematics. Everyday operations such as opening a current account, subscribing to a pension plan or obtaining a mortgage loan require this type of mathematics.

Beyond the biological context, our daily lives are also immersed in a physical environment. Measuring magnitudes such as temperature and speed is essential. The constructions that surround us, such as buildings, roads, bridges and squares, present opportunities to analyse geometric shapes. The development of these structures requires calculations, measurements, estimations and statistical analysis.

Weather phenomena provide excellent examples of random events. The duration, intensity, and extent of rain, thunderstorms, and hail, as well as maximum and minimum temperatures and wind intensity and direction, are all random variables. The consequences of these phenomena, such as the volume of water in a swamp or the extent of damage caused by flooding or hail, also offer opportunities to study statistics and probability.

Mathematical culture

Placing problem solving and modelling at the forefront of mathematics education has important educational implications. It would be contradictory to present mathematics as something closed, disconnected from reality and complete, considering its historical origins and current applications. It is crucial to recognise that mathematical knowledge enables us to model and solve problems from different fields, while problems originating in non-mathematical contexts often form the intuitive basis for the development of new mathematical knowledge.

Education aims at developing well-rounded citizens, and with the evolution of the concept of culture in modern society, the role of mathematics is gaining recognition. Mathematics education aims to cultivate this cultural aspect. However, the aim is not to transform future citizens into "amateur mathematicians" or to train them in complex calculations, as computers can now perform such tasks (Gravemeijer, 1997). Instead, the aim is to equip people with the ability to critically interpret and evaluate mathematical information and arguments supported by data found in various contexts, such as the media or their professional work.

Furthermore, mathematics education aims to improve communication skills, enabling people to discuss and transmit mathematical information effectively, as well as to solve mathematical problems encountered in daily life or in professional settings. From a pedagogical and epistemological perspective, it is crucial to differentiate between the process of constructing mathematical knowledge and the characteristics of well-developed mathematical knowledge.

Formalization, precision, and lack of ambiguity are traits associated with mathematics as a mature science. However, the construction of mathematical knowledge, both historically and in the process of individual learning, is inseparable from concrete activities, intuition, and inductive approaches activated through tasks and problem solving. Gaining experience and understanding of mathematical notions, properties, and relationships through real-world applications is not only a prerequisite for formalization, but also a necessary condition for correctly interpreting and utilizing the potential within formal mathematical structures (Mosterín, 1987).

When it comes to teaching mathematics, it is crucial to adapt the approach to the age and knowledge level of the students. Different individuals have dissimilar needs and perceptions of the physical and social environment, as well as different interests compared to adults. Therefore, mechanically transferring "real" situations, even if meaningful to adults, may not capture students' interest.

The historical construction of mathematics highlights the importance of empirical-inductive reasoning, which often plays a more active role in developing new concepts compared to deductive reasoning. Mathematicians do not formulate theorems on the first attempt. Instead, they rely on previous trials, examples and counterexamples, specific case solutions, and the ability to modify initial conditions to investigate outcomes. These intuitive processes provide invaluable clues for developing propositions and theories. Neglecting these intuitive procedures, as observed in some curricular proposals, deprives students of a powerful tool for exploring and constructing mathematical knowledge (English, 2008).

Language and communication

Mathematics, like other scientific disciplines, encompasses a body of knowledge that has its own unique characteristics and internal organization. What distinguishes mathematical knowledge is its incredible ability to be communicated concisely and unambiguously. This is made possible by using various systems of symbolic notation, such as numbers, letters, tables, and graphs. By using these tools, mathematics can accurately represent a wide range of information, drawing attention to aspects and relationships that may not be easily observable.

Thus, mathematics enables us to anticipate and predict future events, situations, or outcomes that have not yet occurred. Take, for example, the expression "2n" to represent an even number. This simple equation is equivalent to (n+1) +(n-1), which reveals that every even number can be expressed as the sum of two consecutive odd numbers. However, it would be a mistake to assume that the power of mathematical knowledge lies solely in its precise and unambiguous symbolic notations. The ability of these notations to represent, explain, and predict is deeply rooted in mathematical knowledge itself, and the notations function as a supporting framework.

The emphasis on the need for constructive activity should not lead to ignoring the internal structure of mathematics, which serves to connect and organize its various components. Indeed, the structure of mathematics is particularly complex and meaningful. This structure has a vertical aspect, where certain concepts build on others, creating a temporal sequence in learning.

Sometimes it requires working on certain aspects only to integrate others, which are considered more important from an educational point of view. However, it is important to note that there is rarely a single or clearly superior path, and if there is, it is usually based on pedagogy rather than epistemology. On the contrary, certain conceptions about the internal structure of mathematics can hinder learning, as demonstrated by the attempt to base all school mathematics on set theory.

Another implication of the relational nature of mathematics is the existence of general strategies or procedures that can be applied in different fields and for different purposes. For example, concepts such as number, counting, ordering, classifying, symbolizing, inferring; are equally useful in geometry and statistics. To help students recognize the similarity and usefulness of these strategies and procedures from different perspectives, it is important to carefully select the teaching content and pay special attention to it.

The foundation of logical-mathematical knowledge lies in the human being's ability to establish relationships between objects or situations based on their interaction with them. This ability includes the ability to abstract and consider these relationships above others that may also be present. For example, in the statements "A is larger than B", "A is three centimeters larger than B", "B is three centimeters shorter than A", we are not simply describing properties of the objects A and B themselves.

Instead, we are expressing the relationship between a shared property - size - which is the result of comparing objects in terms of this property, ignoring many other properties such as color, shape, mass, density, volume. Things such as "larger than", "smaller than", "three centimeters more than", "three centimeters less than", are mental constructs rather than direct observations of the object's properties. Even describing objects, A and B as large or small involves comparing them to other similar objects one has encountered in the past.

This simple example illustrates how mathematical knowledge involves the construction of relationships through interaction with objects. Mathematics is therefore more about construction than deduction in terms of its development and acquisition (English & Sriraman, 2010). If we separate mathematical knowledge from its constructive origins, we risk reducing it to pure formalism, losing its potential as a tool for representation, explanation and prediction.

Throughout its historical development, mathematics has revealed an additional characteristic: its ability to provide us with a dual perspective of reality. On the one hand, mathematics is considered an "exact science" where the results of operations and transformations are unambiguous. However, when we compare mathematical models with real-world situations, they are always approximate due to their inability to perfectly represent reality.

While some aspects of this duality may be apparent in students' first encounters with mathematics, others may become apparent later. Unfortunately, many educational curricula tend to prioritize one side of this coin, favoring the traditional view of mathematics as an exact science. This means that concepts such as certainty ("yes" or "no," "true" or "false") are often emphasized over probability ("it is possible that…", "with a significance level of…") and precision ("the diagonal measures 2," "the area of a circle is πr^2") is prioritized over estimation ("I am wrong by at most one-tenth," "the golden ratio is approximately 5/3”). It is crucial that mathematics education embrace both approaches, not only because they provide a wealth of intrinsic value, but also because the neglected side of the dual perspective has significant implications for the practical applications of mathematics in today’s world.

Chapter 4

A realistic mathematics perspective in early childhood education

Traditionally, mathematics education has focused primarily on memorization to get students to solve exercises and pass tests. Unfortunately, this approach has led to a lack of transferability of mathematical knowledge to real-life situations, leaving many people struggling to apply what they learned in school to everyday scenarios where mathematical understanding is crucial.

The consequences of this teaching model are evident in international assessments of mathematical performance. For example, in the TIMMS 2015 study, Spanish students in the fourth grade of Primary Education obtained 505 points, while the OECD average was 525 points. Similarly, in the PISA 2015 study, Spanish results were four points below the OECD average (490 points), according to data from the Ministry of Education, Culture and Sport (2016a, 2016b).

The poor mathematical performance of primary and secondary school students is believed to be partly attributable to the way mathematics is taught in early childhood education. In many cases, the predominant approach to teaching mathematics at an early age still revolves around memorizing basic skills through repetition. To a lesser extent, the conceptual approach is also used, which emphasizes learning procedures with understanding through manipulation and experimentation with materials.

While this approach may occasionally present activities without a clear context or purpose, there is an effort to promote meaningful learning. However, this learning often fails to materialize due to inadequate teacher management. In addition to these approaches, current curriculum guidelines suggest that drawing knowledge from one's own experiences and considering the contexts of daily life is crucial for effective mathematics teaching and learning, especially at an early age.

In this sense, current curricular guidelines indicate that the objective is to involve children in the process of discovering and representing the various contexts that make up their environment, facilitating their integration and participation in them (Bressan et al., 2016). In the field of mathematics education, this perspective aligns closely with the principles of Realistic Mathematics Education (RME) developed by Freudenthal. RME advocates learning mathematics by actively participating in real or realistic contexts, such as everyday life situations, which resonate with students' experiences and understanding.

Realistic mathematics education is based on the belief that mathematics should have practical value, be relatable to children and relevant to society. According to Freudenthal, although not all children become mathematicians, all adults use mathematics to solve everyday problems.

In line with these principles, several authors have described EMR based on the following ideas:

- Activity: Mathematics is a human activity, and its main objective is to organize the world around us through mathematical concepts.
- Reality: Mathematics is best learned in real-world or realistic contexts.

- Levels: Understanding mathematics involves progressing through diverse levels, including the situational level (understanding within a specific situation), the referential level (using models and descriptions), the general level (exploration, reflection, and generalization), and the formal level (using standard procedures and notation).
- Guided Reinvention: Formal mathematical knowledge is developed through the guidance and mediation of the teacher.
- Interaction: Teaching mathematics is considered a social activity, where interaction between students and between students and teachers can enhance understanding.
- Interconnection: Different mathematical topics should not be taught in isolation but rather integrated.

Children should learn mathematics in real and meaningful contexts, which allow them to develop concepts and apply rules. This approach emphasizes the need to transfer problems from everyday life to the realm of mathematics, solve them, and then transfer the solutions back to the real world, thus familiarizing students with the mathematical world.

The measure

From the process of comparison and using an object or one's own body as a unit of measurement, one can begin to quantify a given measurement. This involves the use of familiar and, eventually, conventional anthropometric units. Estimation is also a key aspect of the quantification process. Contrary to widespread belief, estimation is not simply a guess. It involves making value judgments and approximations based on previous information or experience. It requires mental operations and the use of quick and simple numbers.

An estimate may not provide an exact value, but it allows for decision making and people can interpret it differently. According to Alsina (2006), the process of learning about magnitudes with understanding can be divided into three phases:

- The first phase involves preparing for measurement, during which students identify magnitudes through activities that involve comparisons using quantifiers such as "more than," "less than," and "equal to."
- The second phase is quantification of measurement, where units are introduced. Initially, children use units related to their own body, such as spans and steps, before moving on to conventional units of the decimal metric system, such as meters, grams and liters.
- The final phase is the decimal measurement system, which is not normally addressed in early childhood education. In this phase, multiples and submultiples of reference units for different magnitudes are introduced.

Measurement is a branch of mathematics that encompasses knowledge and activities related to continuous quantities and measurable attributes commonly found in everyday life, such as length, area, volume, capacity, mass, and time. It is closely related to geometry as it involves understanding space, as well as numbers and operations, since measurements are expressed using numerical values. Measurement is also strongly linked to our understanding of the natural world.

Clements and Sarrama (2014) propose learning trajectories for early childhood measurement that help define what mathematical concepts children can understand, how they understand them, and how adults can support their understanding. These trajectories describe the developmental progression and the age at which ideas are typically acquired. For example, children can recognize lengths by age 3, make direct and indirect comparisons of lengths by age 4, order lengths sequentially by age 5, and finally measure lengths by age 6.

Similar progressions exist for area, volume and angles. Interestingly, it is often wrongly assumed that the acquisition of measurement knowledge occurs outside of school, as it is often delegated to the family and social environment. However, this learning often does not occur. Unfortunately, measurement is often introduced and taught algorithmically at later stages of primary education, focusing on unit transformations and losing the true meaning and significance of measurement.

Therefore, it is of utmost importance to incorporate systematic instruction in measurement from an early age, encouraging students to interpret and interact with their environment and real-life situations. This includes observing, comparing, and evaluating results to give measurement knowledge true importance and meaning. In addition to these ideas, it is also valuable to consider international curricular guidelines, such as those provided by the National Council of Teachers of Mathematics, the Common Core State Standards Initiative, and current curricular policy guidelines in Spain, to achieve a comprehensive understanding of measurement concepts in early childhood education (Hurford, 2010).

According to Alsina, physical exploration of the attributes of objects is prioritized primarily. This includes activities such as discriminating between different attributes of objects and materials, identifying qualities. These activities help children ages 3 to 6 develop an understanding of attributes in general, particularly measurable attributes. This understanding is primarily achieved through direct experiences in comparing objects, counting units, and making connections between spatial concepts and numbers.

However, these contents focus on qualitative relationships between objects and do not explicitly address magnitudes. Developing interest and curiosity about measuring instruments. In general terms, the curriculum guidelines cover the main aspects that should be considered when teaching measurement to young children. However, they place little emphasis on the measurement process itself, which is the same for any measurable attribute: selecting a unit, comparing it to the object, and recording the number of units. In relation to the examination of these contents, Alsina asserts that it is crucial to recognize the measurable attributes that young children can understand at an early age. These attributes include length, volume, weight, area, and time. It is important to involve children in various comparison activities, introduce them to the concept of quantification in measurement, and gradually expose them to several types of units, starting with anthropomorphic units and then progressing to conventional units. In addition, it is essential to encourage the practice of measurement through both direct and indirect measurement techniques, using appropriate instruments.

The European Higher Education Area has established a system that aims to improve the quality of universities in Europe. In this system, students are the focal point and take responsibility for their own learning. They actively build knowledge by connecting latest ideas with existing ones, while the teacher acts as a facilitator.

This shift in the approach to teaching and learning in universities is not simply the result of discussions between ministers about what students should learn and how, but has its roots in educational research, including research in mathematics education. One specific model that has emerged from mathematics education research is realist learning.

The model, based on Realistic Mathematics Education (RME), aims to provide an active and practical training experience. The theoretical foundations of realistic learning were established through the Comenius Project 2003-2005, led by Professor Ko Melief of Utrecht University. From this project, Esteve, Melief and Alsina (2009) coordinated a book that not only describes the theoretical foundations of realistic learning, but also provides practical tools and techniques for implementing it in university classrooms. The book also includes accounts of teacher training experiences in various fields, including mathematics education. Research in mathematics education, such as the work of Kilpatrick (1992), has played a key role in shaping this original approach. Mathematics education is often considered important in the school curriculum because of its relative independence from external influences, its hierarchical and cumulative nature, its abstract and arbitrary concepts, and the range of complexity it offers in learning tasks.

This theory, however, does not claim to be a comprehensive learning theory like constructivism. It was developed at the Institute for the Development of Mathematics Education at Utrecht University, which is now called the Freudenthal Institute. The most noteworthy features of EMR can be summarized as follows: it uses everyday life situations or contextual problems as a starting point for learning mathematics. The situations are gradually mathematized using models, which act as mediators between the abstract and the concrete, ultimately forming more formal and abstract structures (Gravemeijer & Terwel, 2000).

Thus, EMR emphasizes classroom interaction, both among students and between teacher and students. This intense interaction allows teachers to incorporate students' productions into their teaching. Another key idea of EMR is to give students the opportunity to reinvent mathematics under adult guidance, rather than simply being presented with pre-constructed mathematical concepts.

During its initial stage, EMR was characterized using contexts to bridge the gap between concrete and abstract concepts, the use of models to facilitate progress, and the incorporation of students' free constructions and concepts into the teaching and learning process. The various axes of the mathematics curriculum were also intertwined. EMR initially consisted of basic ideas focused on the how and what of mathematics teaching. Over time, these ideas were accumulated and revised, leading to the formation of EMR as we know it today.

The book “Realistic Learning in Initial Teacher Training” by Esteve, Melief and Alsina, proposes that pre-service teachers should be exposed to diverse ways of acting and actively practice them. They should develop criteria to determine when, what and why certain actions are appropriate, and should engage in systematic reflection. According to this perspective, experiences and practice serve as a basis for professional learning and are essential for pre-service teachers to align theory, classroom practice and their own personal characteristics.

The first principle emphasizes the importance of starting with real classroom experiences and problems faced by preservice teachers. Instead of introducing theoretical concepts from textbooks, training begins by actively engaging preservice teachers in specific classroom situations and encouraging them to reflect on their thoughts, feelings, needs, and interests. This inductive approach aims to establish a connection between these experiences and preservice teachers' future professional roles (Parra, 2013).

The second principle focuses on promoting systematic reflection. It suggests that learning from experiences is a natural process, but it must be guided intentionally. Here, five phases are identified in this reflective process: action or experience, looking back at the action, becoming aware of important aspects of one's own performance, seeking alternative behaviors, and testing their effectiveness in new situations. Each cycle of reflection begins with a new experience and contributes to professional growth.

The third principle highlights the social and interactive nature of learning. Group discussions and interactions among preservice teachers are crucial to promoting reflection. By sharing their experiences, preservice teachers are encouraged to structure their thoughts, explore different perspectives, and receive feedback from their peers. These reflective interactions deepen the intentional purpose of professional learning and contribute to collective reflection and construction.

The fourth principle distinguishes three levels of learning in teacher education: representation, schema, and theory. At the representation level, preservice teachers react spontaneously based on their needs, values, opinions, feelings, and unconscious tendencies. At the schema level, reflection occurs during or after a situation, leading to the development of concepts, characteristics, and principles to describe practice. The theoretical level involves constructing a logical order and analyzing conceptual relationships within an individual schema or connecting multiple schemas to form a coherent theory. Realistic learning in teacher education involves co-construction of knowledge by incorporating preservice teachers' existing knowledge and experiences with new knowledge and skills provided by teachers, colleagues, and other resources. It promotes systematic reflection, social interaction, and the development of a personal and professional identity.

The fifth principle emphasizes the importance of considering pre-service teachers as individuals with their own identity. Autonomy and self-regulated professional development are crucial for their growth. Developing self-awareness and fostering interest in one's own identity is essential for pre-service teachers to develop their potential and transfer it to others. This perspective highlights the need to integrate a moral foundation into training by helping pre-service teachers develop their own identity.

The development of professional skills goes beyond the mere display of knowledge and abilities. A competent professional also recognizes the importance of aligning their actions with their knowledge, skills, motivations, and values. They approach problem-solving with flexibility, dedication, and perseverance. Therefore, the acquisition of reflective competence during initial teacher education is of utmost importance. Students must cultivate a deep understanding of the need for and commitment to act in accordance with their professional expertise to effectively address challenges that arise in their future practice.

Both individual and group reflection are crucial in the process of training new teachers. Rather than prescribing solutions, collaborative support aims to help student teachers navigate their own learning journey. This involves encouraging them to express their concerns and needs, listening to their perspectives, drawing on their existing knowledge, and providing them with guidance and advice. This type of support also ensures a gradual transfer of control over the learning process to the students themselves.

Peer collaboration has been recognized as a powerful tool for enhancing and developing higher cognitive processes in learning. Recently, there has been a growing interest in Vygotskian approaches that focus on collective scaffolding, which involves the co-construction of knowledge among teachers from each member's contributions through interactions within a group of students or novice teachers. This collaborative work, when well guided and supported, facilitates reflective processes as each learner, including pre-service teachers, verbalizes their internal thoughts and ideas about the world and their environment.

Self-regulation is another important aspect of teacher training. Students must learn to reflect on their own actions, confront their own realities, and identify and solve their own problems. By engaging in ongoing reflection on their daily work, they develop the ability to independently identify areas for improvement and find solutions. This dimension of self-regulation is crucial to fostering autonomous learning and is based on observation, critical analysis, and self-assessment. To help students achieve this level of autonomy, they must be equipped with appropriate tools, such as portfolios and metacognitive prompts.

Conclusion

In his work, Freudenthal introduces the term "phenomenology" to describe his method of analyzing mathematical concepts. This method involves analyzing the phenomena that organize mathematical concepts, considering their development and current use. Freudenthal distinguishes several types of phenomenology, such as pure, didactic, genetic, and historical phenomenology, each focusing on various aspects of the concept studied. A solid analysis of pure phenomenology is crucial to support any effective phenomenological analysis in mathematics teaching.

Mathematical concepts are means of organizing phenomena in the physical, everyday world, and are created through the relationship between phenomena and the means of organization described by mathematical sign systems. These mathematical signs are part of a mathematical sign system containing diverse expressions, and they create abstract concepts as the means of organization become more abstract. Mathematical objects are incorporated into our experiential world, becoming phenomena in a new relationship with other phenomena and means of organization.

Mathematics therefore exists within the same world as the phenomena it organizes, and not in a separate world. Mathematical concepts do not exist as pre-existing ideal objects but are created through mathematical activity and mathematical sign systems. Finally, mathematical concepts are subject to modification over time due to their use and to the new mathematical sign systems in which they are described, but these modifications do not indicate errors in the original concepts or a linear progression toward a single truth.

Problem solving is not the only mathematical activity that creates concepts. Organizing the results obtained from solving problems and proving theorems into a deductive system is another important aspect of mathematics. This systematic organization can take many forms: from local to global, from axiomatic to formalized. However, it is a fundamental element of mathematics because mathematicians have moved from collecting results and methods to developing complex structures. During this process, the use of definitions in mathematics also developed. In mathematics, definitions are not only used to explain to people the meaning of terms, but they are also a link in the chain of inference in organizing the inference system.

The process of definition involves organizing the properties of a mathematical object by inference. The focus is on identifying properties that can be used to create a deductive system, both at a local and global scale, in which a mathematical object can be embedded. It is important to note that isolating specific properties to define a concept is not a neutral or harmless action. In conclusion, this suggests that concepts were originally created to organize related phenomena and means that the content of the concept is now determined by the conclusions drawn within the defined system. Just like theorem proving, the process of definition also leads to the formation of new concepts.

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